

NOTE ON HESSE'S GENERALIZATION OF PASCAL'S THEOREM.

BY THOMAS MUIR, LL.D.

(Read September 16, 1914.)

1. Having appended an additional column and an additional row, namely,

$$\alpha_1, \alpha_2, \dots, \alpha_n, 0 \quad \text{and} \quad \beta_1, \beta_2, \dots, \beta_n, 0$$

to the matrix of the determinant $|u_{1n}|$, Hesse* denoted by $[\alpha, \beta]$ the determinant of the matrix thus resulting; and if the matrix of $[\alpha, \beta]$ were similarly bordered by

$$\gamma_1, \gamma_2, \dots, \gamma_n, 0, 0 \quad \text{and} \quad \delta_1, \delta_2, \dots, \delta_n, 0, 0$$

the determinant of the resulting matrix he denoted by

$$[\alpha\gamma, \beta\delta].$$

With the help of this notation a number of interesting identities were then established, and these he utilized for the purpose of proving an important extension of Pascal's theorem.

2. The last of the identities reached by him was

$$\begin{aligned} |u_{1n}| \cdot \{ [\epsilon\zeta] [\alpha\gamma, \beta\delta] + [\epsilon, \beta] [\alpha\gamma, \delta\zeta] + [\epsilon, \delta] [\alpha\gamma, \zeta\beta] \} \\ = \begin{vmatrix} [\alpha, \beta] & [\alpha, \delta] & [\alpha, \zeta] \\ [\gamma, \beta] & [\gamma, \delta] & [\gamma, \zeta] \\ [\epsilon, \beta] & [\epsilon, \delta] & [\epsilon, \zeta] \end{vmatrix}; \end{aligned} \tag{Z}$$

and, it being important for him to know whether the cofactor of $|u_{1n}|$ on the left was equal to 0, he satisfied himself that it was so when n is 1, and merely added that the same mode of proof would *not* suffice when $n > 1$. Something more definite than this being desirable, the present note takes the matter up at the point where Hesse left it.

* *Abhandl. . . . bayer. Akad. d. Wiss.*, xi. (1872), pp. 177-192.

In order to make a fair start the reader may assure himself of the truth of (Z) by observing that the cofactor of $[\epsilon, \zeta]$ on the right is

$$\begin{vmatrix} [\alpha, \beta] & [\alpha, \delta] \\ [\gamma, \beta] & [\gamma, \delta] \end{vmatrix};$$

that this is a two-line minor of the adjugate of $[\alpha\gamma, \beta\delta]$; that, further, it is that particular minor which corresponds in position with the minor of zero-elements in $[\alpha\gamma, \beta\delta]$: and that therefore by a theorem of Jacobi's it is equal to $[\alpha\gamma, \beta\delta]$ multiplied by the complementary of the said minor of zero-elements, namely, by $|u_{1n}|$: and, finally, that this agrees with what we see on the left.

3. The cofactor of $|u_{1n}|$ in (Z) is readily recognized to be a sum of products of pairs of determinants, the first factor of each product being of the $(n+1)$ th order and the second factor of the $(n+2)$ th order. This suggests the possibility of the cofactor being representable as a determinant of the $(2n+3)$ th order; and without much difficulty such is found to be the case. Thus, taking $n=4$ merely for shortness' sake in writing, and keeping an eye on the first of its three product-terms, namely,

$$[\epsilon, \zeta] \cdot [\alpha\gamma, \beta\delta],$$

we form a determinant of the 11th order having $[\epsilon, \zeta]$ for its first five-line coaxial minor and having $[\alpha\gamma, \beta\delta]$ for its last six-line coaxial minor, the only change in form being that the row of β 's and the row of δ 's in the latter are transferred from the end to the beginning. Turning next to the second product, namely,

$$[\epsilon, \beta] \cdot [\alpha\gamma, \delta\zeta],$$

we see at a glance that not only it but also $[\epsilon, \delta] \cdot [\alpha\gamma, \zeta\beta]$ is provided for by merely appending to the five-line coaxial minor a row of β 's and a row of δ 's, prefixing to the six-line coaxial minor a row of ζ 's, and filling all the remaining vacant places with zeros. The result of this is

$$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 & \epsilon_1 & . & . & . & . & . & . \\ v_1 & v_2 & v_3 & v_4 & \epsilon_2 & . & . & . & . & . & . \\ w_1 & w_2 & w_3 & w_4 & \epsilon_3 & . & . & . & . & . & . \\ x_1 & x_2 & x_3 & x_4 & \epsilon_4 & . & . & . & . & . & . \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & . & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & . & . \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & . & \beta_1 & \beta_2 & \beta_3 & \beta_4 & . & . \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & . & \delta_1 & \delta_2 & \delta_3 & \delta_4 & . & . \\ . & . & . & . & . & u_1 & u_2 & u_3 & u_4 & \alpha_1 & \gamma_1 \\ . & . & . & . & . & v_1 & v_2 & v_3 & v_4 & \alpha_2 & \gamma_2 \\ . & . & . & . & . & w_1 & w_2 & w_3 & w_4 & \alpha_3 & \gamma_3 \\ . & . & . & . & . & x_1 & x_2 & x_3 & x_4 & \alpha_4 & \gamma_4 \end{vmatrix}.$$

By adding the first four rows one by one in order to the last four rows, and then subtracting the 6th, 7th, 8th, 9th columns from the first four columns in similar fashion, this is readily transformed into

$$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 & \epsilon_1 & . & . & . & . & . & . \\ v_1 & v_2 & v_3 & v_4 & \epsilon_2 & . & . & . & . & . & . \\ w_1 & w_2 & w_3 & w_4 & \epsilon_3 & . & . & . & . & . & . \\ x_1 & x_2 & x_3 & x_4 & \epsilon_4 & . & . & . & . & . & . \\ . & . & . & . & . & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & . & . \\ . & . & . & . & . & \beta_1 & \beta_2 & \beta_3 & \beta_4 & . & . \\ . & . & . & . & . & \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_4 & . & . \\ . & . & . & . & \epsilon_1 & u_1 & u_2 & u_3 & u_4 & a_1 & \gamma_1 \\ . & . & . & . & \epsilon_2 & v_1 & v_2 & v_3 & v_4 & a_2 & \gamma_2 \\ . & . & . & . & \epsilon_3 & w_1 & w_2 & w_3 & w_4 & a_3 & \gamma_3 \\ . & . & . & . & \epsilon_4 & x_1 & x_2 & x_3 & x_4 & a_4 & \gamma_4 \end{vmatrix},$$

which is seen to break up into

$$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix} \cdot \begin{vmatrix} . & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & . & . \\ . & \beta_1 & \beta_2 & \beta_3 & \beta_4 & . & . \\ . & \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_4 & . & . \\ \epsilon_1 & u_1 & u_2 & u_3 & u_4 & a_1 & \gamma_1 \\ \epsilon_2 & v_1 & v_2 & v_3 & v_4 & a_2 & \gamma_2 \\ \epsilon_3 & w_1 & w_2 & w_3 & w_4 & a_3 & \gamma_3 \\ \epsilon_4 & x_1 & x_2 & x_3 & x_4 & a_4 & \gamma_4 \end{vmatrix}.$$

Further, the seven-line determinant here, by having its first column made the last and its 1st, 2nd, 3rd rows the 7th, 5th, 6th respectively is seen to be the bordered determinant which Hesse would have denoted by

$$[a\gamma\epsilon, \beta\hat{c}\zeta].$$

Our final result thus is

$$[\epsilon, \zeta] \cdot [a\gamma, \beta\hat{c}] + [\epsilon, \beta] \cdot [a\gamma, \hat{c}\zeta] + [\epsilon, \hat{c}] \cdot [a\gamma, \zeta\beta] = |u_{1n}| \cdot [a\gamma\epsilon, \beta\hat{c}\zeta] \quad (I)$$

4. By transposition of the column of ϵ 's with the column of a 's and the changing of signs throughout, (I) takes the form

$$|u_{1n}| \cdot [a\gamma\epsilon, \beta\hat{c}\zeta] = [a, \beta] \cdot [\gamma\epsilon, \hat{c}\zeta] + [a, \hat{c}] \cdot [\gamma\epsilon, \beta\zeta] + [a, \zeta] \cdot [\gamma\epsilon, \beta\hat{c}],$$

and is then seen to be the second of a series of identities of which Hesse gave the first, namely,

$$|u_{1n}| \cdot [a\gamma, \beta\hat{c}] = [a, \beta] \cdot [\gamma, \hat{c}] - [a, \hat{c}] \cdot [\gamma, \beta].$$

Further, it should be noted how the dropping of ϵ and ζ in the second makes the first a quasi-deduction from it.

Taking an additional column of η 's and an additional row of θ 's we doubtless would find that

$$\begin{aligned} |u_{1n}| \cdot [\alpha\gamma\epsilon\eta, \beta\hat{c}\zeta\theta] &= [\alpha, \beta] \cdot [\gamma\epsilon\eta, \hat{c}\zeta\theta] - [\alpha, \hat{c}] \cdot [\gamma\epsilon\eta, \beta\zeta\theta] \\ &\quad + [\alpha, \zeta] \cdot [\gamma\epsilon\eta, \beta\hat{c}\theta] - [\alpha, \theta] \cdot [\gamma\epsilon\eta, \beta\hat{c}\zeta] \end{aligned}$$

5. Returning now to Hesse's identity (Z) and using the result of § 3 we obtain

$$|u_{1n}|^2 \cdot [\alpha\gamma\epsilon, \beta\hat{c}\zeta] = \begin{vmatrix} [\alpha, \beta] & [\alpha, \hat{c}] & [\alpha, \zeta] \\ [\gamma, \beta] & [\gamma, \hat{c}] & [\gamma, \zeta] \\ [\epsilon, \beta] & [\epsilon, \hat{c}] & [\epsilon, \zeta] \end{vmatrix} \quad (\text{II})$$

This is the second identity of another series, the first of which is

$$|u_{1n}| \cdot [\alpha\gamma, \beta\hat{c}] = \begin{vmatrix} [\alpha, \beta] & [\alpha, \hat{c}] \\ [\gamma, \beta] & [\gamma, \hat{c}] \end{vmatrix},$$

the first identity of both series being in fact the same. The third identity, we may be sure, is

$$|u_{1n}|^3 \cdot [\alpha\gamma\epsilon\eta, \beta\hat{c}\zeta\theta] = \begin{vmatrix} [\alpha, \beta] & [\alpha, \hat{c}] & [\alpha, \zeta] & [\alpha, \theta] \\ [\gamma, \beta] & [\gamma, \hat{c}] & [\gamma, \zeta] & [\gamma, \theta] \\ [\epsilon, \beta] & [\epsilon, \hat{c}] & [\epsilon, \zeta] & [\epsilon, \theta] \\ [\eta, \beta] & [\eta, \hat{c}] & [\eta, \zeta] & [\eta, \theta] \end{vmatrix};$$

and by way of verification we note that the expanding of the determinant on the right in terms of the elements of its first row and their complementaries gives

$$\begin{aligned} &[\alpha, \beta] \cdot \begin{vmatrix} [\gamma, \hat{c}] & [\epsilon, \zeta] & [\eta, \theta] \end{vmatrix} - [\alpha, \hat{c}] \cdot \begin{vmatrix} [\gamma, \beta] & [\epsilon, \zeta] & [\eta, \theta] \end{vmatrix} \\ &+ [\alpha, \zeta] \cdot \begin{vmatrix} [\gamma, \beta] & [\epsilon, \hat{c}] & [\eta, \theta] \end{vmatrix} - [\alpha, \theta] \cdot \begin{vmatrix} [\gamma, \beta] & [\epsilon, \hat{c}] & [\eta, \zeta] \end{vmatrix}, \end{aligned}$$

which by the use of (II) becomes

$$\begin{aligned} |u_{1n}|^2 \cdot \{ &[\alpha, \beta] \cdot [\gamma\epsilon\eta, \hat{c}\zeta\theta] - [\alpha, \hat{c}] \cdot [\gamma\epsilon\eta, \beta\zeta\theta] \\ &+ [\alpha, \zeta] \cdot [\gamma\epsilon\eta, \beta\hat{c}\theta] - [\alpha, \theta] \cdot [\gamma\epsilon\eta, \beta\hat{c}\zeta] \} \end{aligned}$$

and thus reproduces the identity at the end of § 4.

6. The four-line determinant

$$\begin{vmatrix} [\alpha, \beta] & [\gamma, \hat{c}] & [\epsilon, \zeta] & [\eta, \theta] \end{vmatrix}$$

expanded in the preceding paragraph is manifestly a compound determinant with elements of the $(n+1)$ th order, and it is not difficult to see

that the said elements are minors of $[\alpha\gamma\epsilon\eta, \beta\delta\zeta\theta]$, which is a determinant of the $(n+4)$ th order. Further, a little examination shows that they are the minors in that determinant which are complementary to the principal minors of the array

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. \end{array}$$

It is therefore interesting to inquire what would happen if we made the bordered determinant $[\alpha\gamma\epsilon\eta, \beta\delta\zeta\theta]$ a perfectly general determinant, substituting

$$\begin{array}{cccc} u_{1,n+1} & \dots & u_{1,n+4} \\ \dots & \dots & \dots \\ u_{n,n+1} & \dots & u_{n,n+4} \end{array}$$

for the bordering columns, and

$$\begin{array}{ccc} u_{n+1,1} & \dots & u_{n+1,n+4} \\ \dots & \dots & \dots \\ u_{n+4,1} & \dots & u_{n+4,n+4} \end{array}$$

for the bordering rows and the array of zeros. The left-hand side of the identity would then be

$$|u_{1n}|^3 \cdot |u_{11}u_{22} \dots u_{nn}u_{n+1,n+1} \dots u_{n+4,n+4}|,$$

and the four-line compound determinant on the right would be

$$\left| \begin{array}{cccc} |u_{11} \dots u_{nn}u_{n+1,n+1}| & \dots & |u_{11} \dots u_{nn}u_{n+1,n+4}| \\ \dots & \dots & \dots \\ |u_{11} \dots u_{nn}u_{n+4,n+1}| & \dots & |u_{11} \dots u_{nn}u_{n+4,n+4}| \end{array} \right|.$$

On inspection of these we readily perceive not only that the identity still holds, but that it may be viewed as simply an "extensional" of the truism

$$|u_{n+1,n+1} \dots u_{n+4,n+4}| = \left| \begin{array}{ccc} u_{n+1,n+1} & \dots & u_{n+1,n+4} \\ \dots & \dots & \dots \\ u_{n+4,n+1} & \dots & u_{n+4,n+4} \end{array} \right|.$$

The same "extensional" will be found arrived at by a different path in 1881 on page 4 of vol. xxx. of the *Transac. R. Soc. Edinburgh*.

CAPETOWN, S.A.

August 25, 1915.